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Thompson's generalized characters and permutation characters

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ABSTRACT

Given a property $\mathcal{P}$ of groups and a finite group G (not necessarily having this property) J.G. Thompson (1996) [5] defined an associated counting function $\chi_{\mathcal{P}}$ on G. For certain properties $\mathcal{P}$ he then establishes that $\chi_{\mathcal{P}}$ is a generalized character of G. We prove here that, under mild conditions on $\mathcal{P}$, these functions are not only generalized characters but in fact lie in the permutation character ring of G.

1. Introduction

If G is any finite group and $\mathcal{P}$ is any property of groups (or any collection of subgroups of G) define for each $g \in G$ the subset $S_{\mathcal{P}}(g) = \{x \mid \langle g, x\rangle \in \mathcal{P}\}$ and set $\chi_{\mathcal{P}}(g) = |S_{\mathcal{P}}(g)|$. We note that the sets $S_{\mathcal{P}}(g)$ are not generally subgroups of G. However, the counting function $\chi_{\mathcal{P}}$ is certainly a class function on G in case $\mathcal{P}$ is a property of groups as we presume that group properties are preserved by isomorphisms. In case $\mathcal{P}$ is a collection of subgroups of G, we need to further assume that this collection is closed under conjugation to guarantee that $\chi_{\mathcal{P}}$ is a class function.

When $\mathcal{P}$ is the property of having abelian Sylow p-subgroups (where p is a fixed prime), Thompson [5] establishes that $\chi_{\mathcal{P}}$ is a generalized character of G, and Moreto [4] observes that this generalized character need not be a character. Thompson also observes that his argument proving that $\chi_{\mathcal{P}}$ is a generalized character also works for other group theoretic properties as well (nilpotent, solvable ...). Since $\chi_{\mathcal{P}}$ is always integer-valued, it seems reasonable to anticipate that this function lies in the permutation character ring of G. We prove under mild conditions on $\mathcal{P}$ that this is indeed the case.

Some restriction on $\mathcal{P}$ is necessary for $\chi_{\mathcal{P}}$ to be a generalized character. For example, if $\mathcal{P}$ is the property of "being cyclic", and G is a non-cyclic elementary abelian p-group for some prime p, then the inner product of $\chi_{\mathcal{P}}$ with the principal character is not an integer.

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Thompson's argument involves the observation that $\chi_{\mathcal{P}}$ is constant on elements generating the same cyclic subgroup, together with the fact that the cardinalities $|S_{\mathcal{P}}(g)|$ are divisible by $|\mathbf{C}_G(g)|$. Both of these facts together are enough to easily conclude that $\chi_{\mathcal{P}}$ is a generalized character. The clever part of the argument involves showing divisibility of $|S_{\mathcal{P}}(g)|$ by $|\mathbf{C}_G(g)|$ since $S_{\mathcal{P}}(g)$ is not generally a union of cosets (right or left) of $\mathbf{C}_G(g)$. Instead, Thompson uses a different decomposition of G into subsets, each of size $|\mathbf{C}_G(g)|$, in such a way that $S_{\mathcal{P}}(g)$ is a union of some of these. These "symmetric cosets" of $\mathbf{C}_G(g)$, sometimes referred to somewhat ingenuously as "fake cosets", are the sets $E(x)$ appearing below in Definition 3.1. No particular properties of $\mathbf{C}_G(g)$ are needed to construct this partition of G, and indeed, symmetric cosets exist for any subgroup of G not just centralizers. The term "symmetric" seems apt since these sets are closed under conjugation by elements of the subgroup. (Incidentally, the use of symmetric cosets provides an amusing, if somewhat cumbersome, alternate proof of Lagrange's Theorem.)

Symmetric cosets have been used much earlier by R. Brauer [1] to give a purely group theoretic proof of the celebrated theorem of Frobenius on counting solutions to $x^n \in \mathcal{K}$ in groups, where $\mathcal{K}$ is any conjugacy class. (Indeed, Brauer admits to having found the argument while he was still a student.) In Brauer's paper, two equivalence relations are defined on a group G with subgroup H: "equivalence with regard to H" and "weak equivalence with regard to H", the former being a refinement of the latter. In the first case the equivalence classes are the sets $xD(x)$ and in the second they are the symmetric cosets $E(x)$, both defined below in Definition 3.1, using H in place of C in that definition. Frobenius' Theorem follows directly from Brauer's observation that if H is any subgroup of G whose order divides n, then the solutions to $x^n \in \mathcal{K}$ are a union of symmetric cosets of H.

In this paper G always denotes a finite group, and we prefer to regard $\mathcal{P}$ as a collection of subgroups of G. Call two subgroups H_1 and H_2 of G **linked** if there exists a subgroup $N \leqslant H_1 \cap H_2$ and elements x_i normalizing N so that $H_1 = \langle N, x_1\rangle$, $H_2 = \langle N, x_2\rangle$ and $x_1x_2^{-1} \in \mathbf{C}_G(N)$. This last condition is of course equivalent to saying that x_1 and x_2 induce the same automorphism on N. The terminology **linked** also appears with this same definition in some unpublished notes by Isaacs [2] on this subject. The following definition establishes the conditions on $\mathcal{P}$ that will be needed.

Definition 1.1. A collection of subgroups $\mathcal{P}$ of a finite group G is *admissible* if:

1. $\mathcal{P}$ is closed under conjugation.
2. If $H_1 \leqslant G$ is linked to $H_2 \leqslant G$ then $H_1 \in \mathcal{P}$ if and only if $H_2 \in \mathcal{P}$.

Clearly, any group theoretic property of a subgroup is preserved by conjugation. Although linked subgroups need not be isomorphic, they do share a common commutator subgroup, and so linkage still preserves many group theoretic properties. For example, the property of having abelian Sylow p-subgroups for some fixed prime p is preserved, as Thompson [5] proves (literally in the last two paragraphs of his paper just before his closing remark). Other properties preserved under linkage are: having isomorphic derived subgroups, being solvable, solvable of derived length 3, nilpotent, etc.

As already noted, the first condition of admissibility implies that $S_{\mathcal{P}}(g^x) = S_{\mathcal{P}}(g)^x$ for all group elements g, x so that the associated counting function $\chi_{\mathcal{P}}$ is a class function on G. The following theorem is (essentially) proved in [5].

Theorem A. *If $\mathcal{P}$ is an admissible collection of subgroups of G then $\chi_{\mathcal{P}}$ is a generalized character of G.*

Thompson actually proves Theorem A for a specific property $\mathcal{P}$, but the only condition needed for the proof to work is admissibility. Theorem A is an immediate corollary of the following result, which is the main theorem of this paper.

Theorem B. *If $\mathcal{P}$ is an admissible collection of subgroups of G then $\chi_{\mathcal{P}}$ is in the permutation character ring of G.*

2. Inducing from subsets

If A is a subset of G that is not necessarily a subgroup and if $f : A \to \mathbb{C}$ is any complex valued function, define the induced function $f^G : G \to \mathbb{C}$ by the formula that is usually associated with inducing class functions (especially characters) from subgroups:

$$f^G(g) = \frac{1}{|A|} \sum_{y \in G} f^0\bigl(ygy^{-1}\bigr).$$

Here f^0 denotes the function defined on G that agrees with f on A, and is zero on $G - A$. If $g \in G$, we also define f^g to be the function defined on A^g by $f^g(a^g) = f(a)$ for all $a \in A$.

As might be expected, several properties of ordinary induction carry over to this slightly more general setting. The following proposition lists several useful properties of this generalization that will be needed. All parts are elementary and its proof is omitted.

Proposition 2.1. *Let G be a finite group and let A and B be subsets of G.*

1. *If $f : A \to \mathbb{C}$ is any function then f^G is a class function on G. Moreover, f^G is a character when A is a subgroup and f itself is a character.*
2. *(Linearity) The assignment $f \mapsto f^G$ is linear.*
3. *(Conjugation) If $g \in G$ and f is defined on $A \subseteq G$ then $(f^g)^G = f^G$.*
4. *(Additivity) If A and B are disjoint subsets of G, $S = A \cup B$ and f is defined on S then $|S| f^G = |A|(f|_A)^G + |B|(f|_B)^G$.*
5. *(Transitivity) If $A \subseteq H \leqslant G$ where H is a subgroup of G, and $f : A \to \mathbb{C}$ is any function, then $(f^H)^G = f^G$.*

If $S \subseteq G$ let 1_S denote the constant function with value 1 on S (so that 1_S is the principal character of S when S is a subgroup). Clearly $1_{S^g} = (1_S)^g$ for all $g \in G$. Any rational valued character, indeed, any rational valued class function that is constant on elements generating the same cyclic subgroup, lies in the rational permutation character ring. (This fact is essentially Artin's Theorem on rational valued characters. See Theorem 5.21 of [3].) The next result gives an explicit representation of this fact for $\chi_{\mathcal{P}}$ when $\mathcal{P}$ is conjugacy class closed and each set $S_{\mathcal{P}}(g)$ is an actual subgroup. In fact the result shows $\chi_{\mathcal{P}}$ is an actual permutation character if additionally each subgroup $S_{\mathcal{P}}(g)$ contains the centralizer $\mathbf{C}_G(g)$. This happens whenever $\mathcal{P}$ is an admissible collection of subgroups (see Corollary 3.2 below).

Theorem 2.2. *If $\mathcal{P}$ is any conjugacy class closed collection of subgroups of G then*

$$\chi_{\mathcal{P}} = \sum_{i=1}^{k} \frac{|S_{\mathcal{P}}(g_i)|}{|\mathbf{C}_G(g_i)|} \bigl(1_{S_{\mathcal{P}}(g_i)}\bigr)^G,$$

where $g_1, \ldots, g_k$ are representatives of the conjugacy classes of G.

Proof. For ease of notation, write $S(g)$ for $S_{\mathcal{P}}(g)$. The conjugacy class closed assumption on $\mathcal{P}$ implies that $S(g^x) = S(g)^x$, and so $|S(g)|$ and $(1_{S(g)})^G$ depend only on the conjugacy class of g. We also make use of the elementary observation that $g \in S(h)$ if and only if $h \in S(g)$. For $x \in G$ then:

$$\begin{aligned}\sum_{i=1}^{k} \frac{|S(g_i)|}{|\mathbf{C}_G(g_i)|} \bigl(1_{S(g_i)}\bigr)^G(x) &= \sum_{g \in G} \frac{|S(g)|}{|G|} \bigl(1_{S(g)}\bigr)^G(x) \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{y \in G} (1_{S(g)})^0\bigl(yxy^{-1}\bigr)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|G|} \sum_{g \in G} \bigl| \bigl\{ y \in G \bigm| yxy^{-1} \in S(g) \bigr\} \bigr| \\
&= \frac{1}{|G|} \sum_{g \in G} \bigl| \bigl\{ y \in G \bigm| g \in S\bigl(yxy^{-1}\bigr) \bigr\} \bigr| \\
&= \frac{1}{|G|} \bigl| \bigl\{ (g, y) \in G \times G \bigm| g \in S(x)^{y^{-1}} \bigr\} \bigr| \\
&= \frac{1}{|G|} \sum_{y \in G} \bigl| S(x)^{y^{-1}} \bigr| \\
&= \bigl| S(x) \bigr| = \chi_{\mathcal{P}}(x). \qquad \square
\end{aligned}
$$

Although it is not generally true that each subset $S_{\mathcal{P}}(g)$ is a subgroup of G, nevertheless, when $\mathcal{P}$ is an admissible family of subgroups of G it is true that each term occurring in the sum appearing in the statement of Theorem 2.2:

$$\frac{|S_{\mathcal{P}}(g)|}{|\mathbf{C}_G(g)|} (1_{S_{\mathcal{P}}(g)})^G$$

lies in the permutation character ring of G. This is the key for proving Theorem B.

3. The permutation character ring and symmetric cosets

We begin with a lemma. In the following φ denotes Euler's totient function.

Lemma C. *Let $G = CA$ where $C \trianglelefteq G$ is cyclic of order n, $A \cap C = 1$ and A acts faithfully on C. Also let $X \subseteq C$ be the set of generators of C (so that $|X| = \varphi(n)$) and define $\theta : G \to \mathbb{Z}$ by*

$$\theta(x) = \begin{cases} n, & x \in X, \\ 0, & \textit{otherwise}. \end{cases}$$

Then θ is a generalized character of G which lies in the permutation character ring of G.

Proof. Since the restriction of a permutation character to a subgroup is again a permutation character of that subgroup, we may enlarge A so that A acts as the full group of automorphisms of C. Write $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where the p_i are distinct primes. Then G is a direct product of groups $G = \prod C_i A_i$ where C_i is a cyclic group of order $p_i^{a_i}$ and A_i acts faithfully on C_i as the full group of automorphisms of C_i (and trivially on C_j for $j \neq i$). Define θ_i on each direct factor $C_i A_i$ analogously to the way θ is defined on G so that if $x \in C_i A_i$ then $\theta_i(x) = p^{a_i}$ if x generates C_i and $\theta_i(x) = 0$ otherwise.

If $k > 1$ then, working inductively, each θ_i is in the permutation character ring of $C_i A_i$. Since $C_i A_i$ is a natural homomorphic image of G, each function θ_i may be pulled back to a function defined on G which we continue to denote by θ_i, and which is still in the permutation character ring, this time of G. Since $\theta = \theta_1 \theta_2 \cdots \theta_k$ it follows that θ is in the permutation character ring of G and we are finished.

Assume then $k = 1$ so that C is cyclic of order $n = p^a$ where p is a prime. Then A acts as the full automorphism group of C so is abelian of order $p^{a-1}(p-1)$. Let B denote the unique subgroup of A of order $p-1$ (so B is trivial when $p = 2$). Also set $H = CB$ and $K = \Phi(C)B$ where $\Phi(C)$ denotes the Frattini subgroup. Notice $H = C$ if $p = 2$, and in any case $K \cap C = \Phi(C)$ by Dedekind's Lemma.

The generalized character $\sigma = 1_H + (1_C)^H - (1_K)^H$ of H is clearly in the permutation character ring of H, and its restriction to C is $\sigma|_C = 1_C + (p-1)1_C - (1_{\Phi(C)})^C = p \cdot 1_C - \rho_{C/\Phi(C)}$ where $\rho_{C/\Phi(C)}$ is the regular character of $C/\Phi(C)$ (viewed as a character of C with kernel $\Phi(C)$). Therefore $\sigma(x) = p$ for $x \in X = C - \Phi(C)$ and $\sigma(x) = 0$ for $x \in \Phi(C)$. When $p = 2$, these values completely determine σ

(as then $H = C$). When $p > 2$, H is a Frobenius group with Frobenius kernel C and complement B, so every element of H outside of C is conjugate to an element of B. Restricting σ to B yields $\sigma|_B = 1_B + (1_{C\cap B})^B - (1_K)^H|_B = 1_B + \rho_B - (1_K)^H|_B$ where ρ_B is the regular character of B.

Now $(1_K)^H = 1_H + \chi$ where χ is the unique nonlinear irreducible character of $H/\Phi(C)$. As χ is induced from a character of C, write $\chi = \lambda^H$ where $1 \neq \lambda \in \mathrm{Irr}(C/\Phi(C))$. Therefore $(1_K)^H|_B = 1_B + (\lambda|_{B\cap C})^B = 1_B + \rho_B$. Hence $\sigma|_B = 1_B + \rho_B - 1_B - \rho_B = 0$. In either case ($p = 2$ or $p > 2$) then, we conclude that σ is supported on X, where it takes on the constant value p.

Since σ is in the permutation character ring of H, it follows that σ^G is in the permutation character ring of G. Moreover, since X is a normal subset of G, σ^G is supported on X and clearly for $x \in X$ it takes on the constant value $\sigma(x) \cdot |G : H| = p \cdot p^{a-1} = p^a$. This now identifies σ^G as θ, completing the proof of the lemma. $\square$

We now define three different collections of subsets of G which are associated with a fixed subgroup C (and these three collections will turn out to be partitions of G). The first two of these were defined and used by Thompson [5] to obtain his generalized characters. The third collection introduced here will be instrumental in showing that these characters in fact lie in the (integral) permutation character ring. In the applications that follow, C will eventually be the centralizer of a fixed element of G. This extra fact about C is not needed for the construction of these subsets and their properties, so in the definition below $C \leqslant G$ is an arbitrary subgroup.

Recall that a *reduced residue system* modulo an integer m is a set of representatives from each congruence class modulo m that is coprime to m.

Definition 3.1. If $C \leqslant G$ is a fixed subgroup, define for each $x \in G$ the integer $\ell(x)$ and subsets $D(x)$, $E(x)$ and $F(x)$ as follows:

(i) $D(x)$ is the largest subgroup of C that is normalized by x. Thus $D(x) = \bigcap_{y\in\langle x\rangle} C^y$ and $D(x) \trianglelefteq \langle x\rangle D(x)$. Also, $\ell(x)$ denotes the order of the cyclic factor group $\langle x\rangle D(x)/D(x)$.
(ii) $E(x) = \bigcup_{c\in C} (xD(x))^c$.
(iii) $F(x) = \bigcup_i E(x^i)$ where the union defining $F(x)$ is taken over any set of integers i that form a reduced residue system modulo $\ell(x)$.

We remark that the notation $D(x)$, $E(x)$ and $F(x)$ does suppress the dependency of these sets on the subgroup C which is used in their construction. This is also true of the function ℓ. This should not cause any confusion in the following however since C will be clear from the context when these objects are used.

The next few lemmas and corollaries establish that the collections $\{xD(x) \mid x \in G\}$, $\{E(x) \mid x \in G\}$ and $\{F(x) \mid x \in G\}$ are all partitions of G, each being a refinement of the next. The sets $E(x)$ are the symmetric cosets of C referred to in the introduction: each has cardinality $|C|$ and is stable under conjugation by C. (See Lemma F (ii) below.)

Much of the following lemma is contained in [5].

Lemma D. *Let C be a fixed subgroup of G, and for $x \in G$, let subsets $D(x)$ and integers $\ell(x)$ be defined as in Definition 3.1 relative to the subgroup C. Then for all $x \in G$ we have $C \cap \langle x\rangle D(x) = D(x)$. Additionally:*

(i) *If $x \in G$ and $c \in C$ then $D(x)^c = D(x^c)$ and $\ell(x^c) = \ell(x)$.*
(ii) *If $x \in G$ and $\gcd(i, \ell(x)) = 1$ then $D(x^i) = D(x)$ and $\ell(x^i) = \ell(x)$.*
(iii) *The sets $xD(x)$ for $x \in G$ are permuted by C and form a partition of G. In particular, if $y \in xD(x)$ then $yD(y) = xD(x)$. Moreover, the function ℓ is constant on the sets $xD(x)$.*

Proof. If $x \in G$ then $D(x) \leqslant C$ and we have by Dedekind's Lemma $C \cap \langle x\rangle D(x) = D(x) \cdot (C \cap \langle x\rangle)$. However, $C \cap \langle x\rangle$ is obviously a subgroup of C that is normalized by x, so must be included in $D(x)$ by definition of $D(x)$. This proves $C \cap \langle x\rangle D(x) = D(x)$.

Since $D(x)$ is the largest subgroup of C normalized by x it follows that $D(x)^c$ is the largest subgroup of $C^c = C$ that is normalized by x^c, that is $D(x)^c = D(x^c)$. Now conjugation by c induces an isomorphism of factor groups $\langle x\rangle D(x)/D(x) \to \langle x^c\rangle D(x^c)/D(x^c)$, and $\ell(x^c) = \ell(x)$ follows, proving (i).

Suppose now i is an integer satisfying $\gcd(i, \ell(x)) = 1$. Since x normalizes $D(x)$ then certainly x^i does as well and we have the inclusion $D(x) \leqslant D(x^i)$. Now $D(x^i)$ is normalized by its subgroup $D(x)$ as well as by x^i and so we have $\langle x^i\rangle D(x) \leqslant \mathbf{N}_G(D(x^i))$. However, the coprime condition on i implies that $\langle x^i\rangle D(x) = \langle x\rangle D(x)$, and since x belongs to this subgroup we conclude that x normalizes $D(x^i)$. By definition of $D(x)$ then, $D(x^i) \leqslant D(x)$ and we have equality $D(x^i) = D(x)$. From the previous equalities of this paragraph we also have

$$\ell\left(x^i\right) = \left|\left\langle x^i\right\rangle D\left(x^i\right)/D\left(x^i\right)\right| = \left|\left\langle x^i\right\rangle D(x)/D(x)\right| = \left|\langle x\rangle D(x)/D(x)\right| = \ell(x)$$

and the rest of (ii) now follows.

From (i) we have for $c \in C$ and $x \in G$ that $(xD(x))^c = x^c D(x^c)$ so the sets $xD(x)$ are permuted by C. Suppose now $y \in xD(x)$. Then $yD(x) = xD(x)$ and so $y \in \langle x\rangle D(x) \subseteq \mathbf{N}_G(D(x))$ so that $D(x)$ is included in the largest subgroup of C normalized by y, which of course is $D(y)$. Therefore $D(x) \leqslant D(y)$. However, $x \in xD(x) = yD(x) \subseteq yD(y)$ and $D(y) \leqslant D(x)$ follows by symmetry. Hence $D(x) = D(y)$ and $xD(x) = yD(x) = yD(y)$. The sets $xD(x)$ therefore partition G and it remains only to prove that the function ℓ is constant on these subsets.

Suppose then $y \in xD(x)$. By the last paragraph we have $D(y) = D(x)$ and $yD(y) = xD(x)$. Clearly, the subgroup generated by $yD(y)$ (and hence $xD(x)$) is $\langle y\rangle D(y) = \langle x\rangle D(x)$ and we have $\ell(y) = |\langle y\rangle D(y) : D(y)| = |\langle x\rangle D(x) : D(x)| = \ell(x)$, and the rest of (iii) follows. □

Corollary E. *The subsets $E(x)$ defined as in Definition* 3.1, *relative to some subgroup C, satisfy the following.*

(i) *If $x \in G$, $\gcd(i, \ell(x)) = 1$ and $i \equiv j \bmod \ell(x)$ then $E(x^i) = E(x^j)$.*
(ii) *The sets $E(x)$ for $x \in G$ partition G.*
(iii) *The function ℓ is constant on the sets $E(x)$.*

Proof. The congruence condition on i and j guarantees that $x^i D(x) = x^j D(x)$ while part (ii) of Lemma D implies $D(x^i) = D(x) = D(x^j)$. Therefore, $x^i D(x^i) = x^j D(x^j)$ and the equality $E(x^i) = E(x^j)$ follows by taking the union of C-conjugates of both sides of the last equation. This proves (i).

By part (iii) of Lemma D we have that the sets $E(x)$ are unions of C-orbits of a partition that is stabilized by C and hence the sets $E(x)$ also partition G, and (ii) follows.

By parts (i) and (iii) of Lemma D, the function ℓ is constant on the sets $E(x)$, completing the proof of the corollary. □

At this point, notice that the sets $F(x)$ (which haven't been used as yet) appearing in Definition 3.1, are well defined because of Corollary E (i).

Because of Lemma D (ii) an alternate description of the sets $F(x)$ is possible. If $x \in G$ is fixed, set $X = \bigcup_i x^i D(x)$, where the integer i runs over a reduced residue system modulo $\ell(x)$. Then $F(x) = \bigcup_{c \in C} X^c$.

Lemma F. *The sets $F(x)$ for $x \in G$ defined relative to the subgroup C, form a partition of G and ℓ is constant on each of these sets. For fixed x, set $A = \mathbf{N}_C(\langle x\rangle D(x))$.*

(i) *$A \cap \langle x\rangle D(x) = D(x)$, A normalizes $D(x)$, and A acts on the factor group $\langle x\rangle D(x)/D(x)$ with kernel $D(x)$. In particular $|A : D(x)|$ divides $\varphi(\ell(x))$.*
(ii) $|E(x)| = |C|$.
(iii) $|F(x)| = |C : A| \cdot \varphi(\ell(x)) \cdot |D(x)| = \frac{\varphi(\ell(x))}{|A:D(x)|} \cdot |C|$.

Proof. If $x \in G$ then $\ell(x^i) = \ell(x)$ whenever $\gcd(i, \ell(x)) = 1$ by Lemma D (ii). It now follows from the definition of $F(x)$ and Corollary E that ℓ is constant on $F(x)$.

Let $\ell = \ell(x)$ be the common value of this function on $F(x)$.

Suppose $y \in F(x)$. Then there exists $c \in C$ and an integer i with $\gcd(i, \ell) = 1$ so that $y \in (x^i D(x^i))^c = (x^i)^c D(x^i)^c = (x^i)^c D((x^i)^c)$. By Lemma D, the sets $zD(z)$ partition G and so we have $yD(y) = (x^i)^c D((x^i)^c)$ as well as $D(y) = D((x^i)^c)$ (as a subgroup is uniquely determined by any of its cosets). Putting all this together:

$$yD(y) = (x^i)^c D\big((x^i)^c\big) = (x^i)^c D(x^i)^c = \big(x^i D(x^i)\big)^c.$$

Now the cosets $yD(y)$ and $(x^i)^c D((x^i)^c)$ are equal elements of the factor groups $\langle y\rangle D(y)/D(y)$ and $\langle (x^i)^c\rangle D((x^i)^c)/D((x^i)^c)$ (which are equal groups). If $\gcd(j, \ell) = 1$, then (because $\ell = \ell(y) = \ell(x^i)$):

$$y^j D\big(y^j\big) = y^j D(y) = \big(yD(y)\big)^j = \big(\big(x^i D\big(x^i\big)\big)^c\big)^j = \big(x^{ij} D\big(x^i\big)\big)^c = \big(x^{ij} D\big(x^{ij}\big)\big)^c,$$

where the first and last equalities follow from Lemma D (ii). Conjugating by $c_1 \in C$ yields

$$\big(y^j D\big(y^j\big)\big)^{c_1} = \big(x^{ij} D\big(x^{ij}\big)\big)^{cc_1}.$$

Now $F(y) = F(x)$ follows by taking the union over all j from a reduced residue system modulo ℓ, and $c_1 \in C$ (since ij will run over a reduced residue system modulo ℓ when j does). This completes the proof that the sets $F(x)$ partition G.

For the rest of the proof we now fix $x \in G$ and set $A = \mathbf{N}_C(\langle x\rangle D(x))$. Notice that A normalizes the group $\langle x\rangle D(x) \cap C = D(x)$, where the equality follows from Lemma D. Therefore $D(x) \trianglelefteq A$, and A acts on the group $\langle x\rangle D(x)/D(x)$. We also note that since $D(x) \leqslant A \leqslant C$ then $D(x) \leqslant A \cap \langle x\rangle D(x) \leqslant C \cap \langle x\rangle D(x) = D(x)$ so that $A \cap \langle x\rangle D(x) = D(x)$.

Let K be the kernel the action of A on $\langle x\rangle D(x)/D(x)$. Then $D(x) \leqslant K$ and $[x, K] \leqslant D(x) \leqslant K$, so $K \leqslant C$ is normalized by x. By definition of $D(x)$ again, we have $K \leqslant D(x)$ so equality holds. Therefore, the group $A/D(x)$ acts faithfully on the cyclic group $\langle x\rangle D(x)/D(x)$ of order $\ell = \ell(x)$, and so it has order $|A : D(x)|$ dividing $\varphi(\ell)$, completing the proof of (i).

Clearly, from its definition, the sets $E(x)$ are stable under conjugation by elements of C. We know from Lemma D (iii) that the sets $xD(x)$ partition G and C permutes these sets. Certainly, $D(x) \leqslant C$ stabilizes $xD(x)$. If $c \in C$ stabilizes $xD(x)$ then c normalizes $\langle x\rangle D(x)$ and so $c \in A$. But c acts trivially on $\langle x\rangle D(x)/D(x)$ as c fixes a generator of this factor group (namely $xD(x)$) and so $c \in D(x)$ by part (i). This identifies the subgroup $D(x)$ as the stabilizer in C of $xD(x)$. It now follows (as the distinct C-conjugates of $xD(x)$ are disjoint) $|E(x)| = |C : D(x)| \cdot |xD(x)| = |C|$, and (ii) follows.

It remains to compute $|F(x)|$.

Set $X = \bigcup_i x^i D(x^i)$ $(= \bigcup_i x^i D(x))$ where the union is taken over a complete set of reduced residues modulo ℓ. The sets in the union are disjoint since these are exactly the $\varphi(\ell)$ generators of the factor group $\langle x\rangle D(x)/D(x)$. Therefore $|X| = \varphi(\ell)|D(x)|$. Also, by definition and the remarks following the proof of Corollary E, $F(x)$ is the union of the C-conjugates of X. The lemma will follow from the fact that distinct C-conjugates of X are disjoint and their number is $|C : A|$. Toward this end it suffices to prove that X is disjoint from each of its distinct C-conjugates, and that the stabilizer (normalizer) of X in C is the subgroup A.

Suppose $c \in C$ satisfies $X \cap X^c \neq \varnothing$. Then there exist integers i and j coprime to ℓ satisfying $x^i D(x^i) \cap (x^j D(x^j))^c \neq \varnothing$. Now $(x^j D(x^j))^c = (x^j)^c D((x^j)^c)$, and since the sets $zD(z)$ form a partition we must have equality $x^i D(x^i) = (x^j D(x^j))^c$. We also have $D(x^i) = D(x^j) = D(x)$ so that $x^i D(x) = (x^j D(x))^c$. The set $x^i D(x)$ generates the group $\langle x\rangle D(x)$, as does $x^j D(x)$. Therefore c normalizes $\langle x\rangle D(x)$ and we have $c \in \mathbf{N}_C(\langle x\rangle D(x)) = A$.

We have already seen that A acts on $\langle x\rangle D(x)$ stabilizing $D(x)$ and hence A acts on the quotient $\langle x\rangle D(x)/D(x)$. Therefore A stabilizes X since X is the union of the cosets which are generators of this factor group. In particular, $X = X^c$. This proves both that the distinct C-conjugates of X are disjoint, and that the stabilizer of X in C is A.

All parts of Lemma F are now proved. $\square$

Thanks to Corollary E (ii) and Lemma F (ii) we can now justify calling the sets $E(x)$ the "symmetric cosets of C": they all have cardinality $|C|$, partition G, and by construction are stable under conjugation by the elements of C.

Recall that if $\mathcal{P}$ is any collection of subgroups of G then $S_{\mathcal{P}}(g) = \{x \mid \langle g, x\rangle \in \mathcal{P}\}$.

Lemma G. *Let $\mathcal{P}$ be an admissible collection of subgroups of G and fix $g \in G$. Set $C = \mathbf{C}_G(g)$, and define for each $x \in G$ the sets $F(x)$ relative to the subgroup C. Then $S_{\mathcal{P}}(g)$ is a union of sets of the form $F(x)$. In particular, $S_{\mathcal{P}}(g)$ is a union of symmetric cosets of $\mathbf{C}_G(g)$.*

Proof. Recall that the subgroup C is also used to define sets $D(x)$ and $E(x)$ for $x \in G$, and that the sets $F(x)$ are defined in terms of these. For ease of notation this dependence on C is suppressed. We also write $S(g)$ for $S_{\mathcal{P}}(g)$.

We know that the sets $F(x)$ partition G and we must show that if $x \in S(g)$ then $F(x) \subseteq S(g)$.

Fix $x \in S(g)$. Then $\langle g, x\rangle \in \mathcal{P}$. Now $D(x) = \bigcap_{y\in\langle x\rangle} C^y = \bigcap_{y\in\langle x\rangle} \mathbf{C}_G(g^y) = \mathbf{C}_G(N)$ where $N = \langle g^y \mid y \in \langle x\rangle\rangle$. Clearly, $\langle g, x\rangle = N\langle x\rangle$ and $N \trianglelefteq N\langle x\rangle$. If $d \in D(x)$ then $x^{-1}(xd) = d \in \mathbf{C}_G(N)$ and so the subgroups $\langle N, x\rangle$ and $\langle N, xd\rangle$ are linked, according to Definition 1.1. Since $\mathcal{P}$ is admissible, this means that $\langle N, xd\rangle \in \mathcal{P}$. However, since x and xd induce the same automorphism of N, the $\langle x\rangle$-conjugates of g are the same as the $\langle xd\rangle$-conjugates of g, and of course these generate N. Hence $\langle g, xd\rangle = \langle N, xd\rangle \in \mathcal{P}$, and this proves $xD(x) \subseteq S(g)$.

Suppose i is an integer satisfying $\gcd(i, \ell(x)) = 1$. Now $\ell(x) = |\langle x\rangle D(x) : D(x)|$ divides $|\langle x\rangle|$ so by elementary number theory we may find an integer $j \equiv i \bmod \ell(x)$ satisfying $\gcd(j, |\langle x\rangle|) = 1$. (The argument is essentially the Chinese Remainder Theorem. Write $|\langle x\rangle| = km$, where k is divisible only by primes dividing ℓ, and $\gcd(m, \ell) = 1$. Then simply choose j to satisfy $j \equiv i \bmod k$ and $j \equiv 1 \bmod m$.) Then $\langle x^j\rangle = \langle x\rangle$ and $x^{i-j} \in D(x)$ so that $x^i D(x) = x^j D(x)$. Now $\langle g, x^j\rangle = \langle g, x\rangle \in \mathcal{P}$ and so $x^j \in S(g)$. We have already seen that this implies $x^j D(x^j) \subseteq S(g)$. By Lemma D (ii) we have $x^j D(x^j) = x^j D(x) = x^i D(x)$. Since this is true for all i prime to $\ell(x)$, we now have $X \subseteq S(g)$ where $X = \bigcup_i x^i D(x)$, the union being taken over a reduced residue system modulo $\ell(x)$. Finally, let $y \in X$ and $c \in C$. Then $y \in S(g)$ so $\langle g, y\rangle \in \mathcal{P}$. Since $\mathcal{P}$ is admissible, it is closed under taking conjugates and hence $\langle g, y^c\rangle = \langle g^c, y^c\rangle = \langle g, y\rangle^c \in \mathcal{P}$. We conclude $X^c \subseteq S(g)$ for all $c \in C$ and so $F(x) = \bigcup_{c\in C} X^c \subseteq S(g)$, as desired. $\square$

We already remarked that the sets $S_{\mathcal{P}}(g)$ are not necessarily subgroups of G. However, if these are subgroups, then the terms appearing in Theorem 2.2 are non-negative rational multiples of permutation characters. If additionally $\mathcal{P}$ is admissible, then these rational coefficients are integers by Lemma G (as each set $S_{\mathcal{P}}(g)$ is then a union of symmetric cosets of $\mathbf{C}_G(g)$). These observations immediately imply the following corollary.

Corollary 3.2. *If $\mathcal{P}$ is an admissible collection of subgroups of G and each set $S_{\mathcal{P}}(g)$ is a subgroup of G (although not necessarily in $\mathcal{P}$), then $\chi_{\mathcal{P}}$ is a permutation character of G.*

4. Proof of Theorem B

Assume $\mathcal{P}$ is an admissible family of subgroups of the group G. Our goal is of course to prove that $\chi_{\mathcal{P}}$ lies in the permutation character ring of G. For ease of notation, write $S(g)$ for $S_{\mathcal{P}}(g)$.

By Theorem 2.2 above, $\chi_{\mathcal{P}}$ is a sum of functions of the form $\frac{|S(g)|}{|\mathbf{C}_G(g)|}(1_{S(g)})^G$ and it suffices to show that each of these functions are in the permutation character ring of G.

Fix $g \in G$ then, and set $C = \mathbf{C}_G(g)$. Using C, construct the sets $D(x)$, $E(x)$, and $F(x)$ for $x \in G$ according to Definition 3.1. By Lemma G, $S(g)$ is partitioned by the sets $F(x)$ which it includes. By additivity of induction (property 4 of Proposition 2.1 above) $\frac{|S(g)|}{|C|}(1_{S(g)})^G$ is a sum of functions of the form $\frac{|F(x)|}{|C|}(1_{F(x)})^G$ so it suffices to prove that these latter functions are in the permutation character ring of G.

Fix x in G and set $X = \bigcup_i x^i D(x)$ where i runs over a reduced residue system modulo $\ell(x)$. Notice that $|X| = \varphi(\ell(x))|D(x)|$. From the remarks following the proof of Corollary E we know that

$F(x) = \bigcup_{c \in C} X^c$, and from the proof of Lemma F, distinct C-conjugates of X are disjoint and their number is $|C : A|$ where $A = \mathbf{N}_C(\langle x \rangle D(x))$. Moreover, for each such conjugate X^c, property 3 of Proposition 2.1 implies $(1_{X^c})^G = ((1_X)^c)^G = (1_X)^G$. Using additivity of induction again we have

$$\frac{|F(x)|}{|C|}(1_{F(x)})^G = \frac{|C:A| \cdot |X|}{|C|}(1_X)^G = \frac{|X|}{|A|}(1_X)^G = \frac{\varphi(\ell(x)) \cdot |D(x)|}{|A|}(1_X)^G.$$

Since A normalizes $\langle x \rangle D(x)$ we may define the subgroup $H = A\langle x \rangle D(x)$. By transitivity of induction and linearity (properties 5 and 2 of Proposition 2.1) we have

$$\frac{\varphi(\ell(x)) \cdot |D(x)|}{|A|}(1_X)^G = \frac{\varphi(\ell(x))}{|A : D(x)|}(1_X)^G = \left(\frac{\varphi(\ell(x))}{|A : D(x)|}(1_X)^H\right)^G$$

and it suffices to show that the function $\psi = \frac{\varphi(\ell(x))}{|A:D(x)|}(1_X)^H$ is in the permutation character ring of H.

Now X is a normal subset of H so ψ is supported on X where it takes on the constant value

$$\begin{aligned}\frac{\varphi(\ell(x))}{|A : D(x)|} \cdot \frac{1}{|X|}|H| &= \frac{\varphi(\ell(x))}{|A : D(x)|} \cdot \frac{1}{\varphi(\ell(x))|D(x)|}|H| \\ &= |H : A| = \big|\langle x \rangle D(x) : D(x)\big| = \ell(x)\end{aligned}$$

where the second to last equality follows from $A \cap \langle x \rangle D(x) = D(x)$, which was part of the conclusion of Lemma F (i).

The group $H/D(x)$ has the structure of the group appearing in Lemma C, where the groups C and A of that lemma are $\langle x \rangle D(x)/D(x)$ and $A/D(x)$ respectively. The function θ appearing in that lemma is in the permutation character ring of $H/D(x)$ and its pullback to H is ψ. This proves that ψ is in the permutation character ring of H, and this concludes the proof of Theorem B. □

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